

GIT-EQUIVALENCE AND DIAGONAL ACTIONS

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ABSTRACT. We describe the GIT-equivalence classes of linearized ample line bundles for the diagonal actions of the linear algebraic groups $SL(V)$ and $SO(V)$ on $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$ and $\mathbb{P}(V)^m$ respectively.

1. INTRODUCTION

Let G be a complex reductive algebraic group, X a projective G -variety, and L an ample G -linearized line bundle over the variety X . Classical Mumford's construction, see [4], takes these objects to the open subset of semi-stable points

$$X_L^{ss} = \{x \in X : F(x) \neq 0 \text{ for some } m > 0 \text{ and } F \in \Gamma(X, L^{\otimes m})^G\}$$

and the categorical quotient $X_L^{ss} \rightarrow X_L^{ss}/G$. Two G -linearized line bundles L_1 and L_2 over the G -variety X are called *GIT-equivalent* if $X_{L_1}^{ss} = X_{L_2}^{ss}$. The papers [3], [6] and [7] are devoted to the study of GIT-equivalence. It is shown that the GIT-equivalence classes define the fan structure on the cone of G -linearized ample line bundles. The main approach used to describe the cones in this fan is the Hilbert-Mumford criterion [4, Chapter 2]; also see [3, Example 3.3.24], where the GIT-equivalence classes are described for the diagonal action of the group $SL(V)$ on the variety $\mathbb{P}(V)^m$.

In [2] an elementary description of GIT-equivalence classes for algebraic torus actions is obtained. The authors use so called orbit cones. Using the Cox construction, in [1] this description is adopted for a large class of G -varieties, compare [7, Section 3]. In [1, Theorem 6.2] there is also a description of the GIT-equivalence classes for the diagonal action of the symplectic group $Sp(V)$ on $\mathbb{P}(V)^m$.

The aim of this paper is to find the GIT-equivalence classes for the diagonal actions of other classical groups. In Section 2 we give some necessary information from works [1] and [2]. In Section 3 we describe the GIT-fan for the diagonal action of the special orthogonal group $SO(V)$, and in Section 4 for the diagonal action of the group $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$. These results are based on the description of generators of the algebra of invariants (The First Fundamental Theorem of the Classical Invariant Theory).

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2. ORBIT CONES AND THE GIT-FAN

Let $G \subseteq GL(V)$ be a complex algebraic group acting diagonally on the space $\mathbb{V} = V^{m_1} \oplus (V^*)^{m_2}$, $m_1 + m_2 = m$. Denote by $P(a_1, \dots, a_m) \subset \mathbb{C}[\mathbb{V}]$, $a_i \in \mathbb{Z}_{\geq 0}$, the subspace consisting of homogeneous polynomials of multidegree (a_1, \dots, a_m) . To each vector $a = (a_1, \dots, a_m) \in \mathbb{Z}_{\geq 0}^m$ assign the open subset

$$U(a) = \{v \in \mathbb{V} \mid \exists k \in \mathbb{N}, F \in P(ka_1, \dots, ka_m)^G : F(v) \neq 0\},$$

where $P(ka_1, \dots, ka_m)^G \subseteq P(ka_1, \dots, ka_m)$ is the subspace of G -invariants. Note that the subset $U(a)$ corresponds to the set of semistable points X_L^{ss} , where $X = \mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$. Here the line bundle L is represented by the point $a \in \mathbb{Z}^m \cong \text{Pic}(X)$.

Two points a and $b \in \mathbb{Z}_{\geq 0}^m$ are called *GIT-equivalent* if $U(a) = U(b)$.

Assume that the algebra of invariants $\mathbb{C}[\mathbb{V}]^G$ is finitely generated and F_1, \dots, F_r are its homogeneous generators. Denote by $a(1), \dots, a(r) \in \mathbb{Z}_{\geq 0}^m$ the multidegrees of F_1, \dots, F_r .

The weight cone $\Omega \subset \mathbb{Q}^m$ is the cone generated by $a(1), \dots, a(r)$.

Lemma 1. *The set $U(a)$ is non-empty if and only if $a \in \Omega$.*

Proof. Suppose $U(a) \neq \emptyset$. Then there exist $k \in \mathbb{N}$ and $F \in P(ka_1, \dots, ka_m)^G$ such that $F \neq 0$. Since $F \in \mathbb{C}[F_1, \dots, F_r]$, we have

$$F = \sum_{p_1, \dots, p_r} c_{p_1 \dots p_r} F_1^{p_1} \dots F_r^{p_r}.$$

If $c_{p_1 \dots p_r} \neq 0$, then we obtain $ka = p_1 a(1) + \dots + p_r a(r)$. Hence $a \in \Omega$.

Conversely, assume that $a \in \Omega$. Then

$$a = \lambda_1 a(1) + \dots + \lambda_r a(r),$$

where $\lambda_1, \dots, \lambda_r \in \mathbb{Q}_{\geq 0}$. Multiplying this equation by the common denominator of $\lambda_1, \dots, \lambda_r$, we get

$$ka = c_1 a(1) + \dots + c_r a(r),$$

where $k, c_1, \dots, c_r \in \mathbb{Z}_{\geq 0}$. Then $F_1^{c_1} \dots F_r^{c_r} \in P(ka)^G$, $F_1^{c_1} \dots F_r^{c_r} \neq 0$. Hence $U(a) \neq \emptyset$. \square

Let $v \in \mathbb{V}$. The orbit cone associated to v is the rational cone

$$\omega(v) = \text{cone}(a \mid \exists F \in P(a)^G : F(v) \neq 0).$$

Lemma 2. *One has $\omega(v) = \text{cone}(a(i) \mid F_i(v) \neq 0)$.*

Proof. It is evident that $\text{cone}(a(i) \mid F_i(v) \neq 0)$ is contained in $\omega(v)$.

Consider the point $a \in \mathbb{Z}_{\geq 0}^m$ and the polynomial

$$F = \sum_{p_1, \dots, p_r} c_{p_1 \dots p_r} F_1^{p_1} \dots F_r^{p_r} \in P(a)^G,$$

such that $F(v) \neq 0$. Then there is a summand $c_{p_1 \dots p_r} F_1^{p_1} \dots F_r^{p_r}$ not vanishing at v . Further, if $p_i \neq 0$, then $F_i(v) \neq 0$. Therefore $a = p_{i_1} a(i_1) + \dots + p_{i_s} a(i_s)$, where $F_{i_l}(v) \neq 0$, $l = 1, \dots, s$. Hence we obtain the inverse inclusion $\text{cone}(a(i) \mid F_i(v) \neq 0) \supseteq \omega(v)$. \square

Corollary. *The set of cones $\{\omega(v) \mid v \in \mathbb{V}\}$ is finite.*

Proposition 1. *Two points a and b are GIT-equivalent if and only if for any $v \in \mathbb{V}$ either $a \in \omega(v)$ and $b \in \omega(v)$, or $a \notin \omega(v)$ and $b \notin \omega(v)$.*

Proof. Suppose $U(a) = U(b)$ and $a \in \omega(v)$. It follows from Lemma 2 that

$$a = \lambda_{i_1} a(i_1) + \dots + \lambda_{i_s} a(i_s),$$

where $F_{i_j}(v) \neq 0$, $j = 1, \dots, s$, $\lambda_j \in \mathbb{Q}_{\geq 0}$. Multiplying this equation by the common denominator of $\lambda_1, \dots, \lambda_r$, we get

$$ka = p_1 a(i_1) + \dots + p_s a(i_s),$$

where $k, p_1, \dots, p_s \in \mathbb{Z}_{\geq 0}$. Then $F_{i_1}^{p_1} \dots F_{i_s}^{p_s} \in P(ka)^G$ does not vanish at v , and hence $v \in U(a) = U(b)$. Therefore there exist $l \in \mathbb{N}$ and $F \in P(lb)^G$ such that $F(v) \neq 0$. Thus $lb \in \omega(v)$ and $b \in \omega(v)$. Similarly if $b \in \omega(v)$, then $a \in \omega(v)$.

It can easily be checked that $a \in \omega(v)$ if and only if $v \in U(a)$. If for any $v \in \mathbb{V}$ either $a \in \omega(v)$ and $b \in \omega(v)$, or $a \notin \omega(v)$ and $b \notin \omega(v)$ hold, then for any $v \in \mathbb{V}$ we have either $v \in U(a)$ and $v \in U(b)$, or $v \notin U(a)$ and $v \notin U(b)$. Hence $U(a) = U(b)$. \square

The GIT-cone of a point $a \in \mathbb{Z}_{\geq 0}^m$ is the cone $\tau(a) = \bigcap_{a \in \omega(v)} \omega(v)$.

Recall that a finite set $\{\sigma_i\}$ of cones in \mathbb{Q}^m is called a *fan*, if

- (1) each face of a cone in $\{\sigma_i\}$ is also a cone in $\{\sigma_i\}$;
- (2) the intersection of two cones in $\{\sigma_i\}$ is a face of each.

Theorem 1. *The set of the cones $\Psi = \{\tau(a) \mid a \in \Omega\}$ is a fan.*

The proof may be found in [2, Theorem 2.11].

The fan Ψ is called the *GIT-fan*. It follows from Proposition 1 that the classes of GIT-equivalence are relative interiors of GIT-cones.

Let $T = (\mathbb{C}^\times)^m$ be a torus. It acts on the space \mathbb{V} as

$$t \circ (v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2}) = (t_1 v_1, \dots, t_{m_1} v_{m_1}, s_1 l_1, \dots, s_{m_2} l_{m_2}),$$

where $t = (t_1, \dots, t_{m_1}, s_1, \dots, s_{m_2}) \in T$, $(v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2}) \in \mathbb{V}$. This action commutes with the action of G , hence the action of T on the categorical quotient $\mathbb{V} // G := \text{Spec } \mathbb{C}[\mathbb{V}]^G$ is well defined.

Consider a point $v \in \mathbb{V}$. It is not hard to see that

$$\dim \omega(v) + \dim T_{\pi(v)} = \dim T,$$

where $\pi : \mathbb{V} \rightarrow \mathbb{V} // G$ is the quotient morphism, and $T_{\pi(v)}$ is the stabilizer of the point $\pi(v)$.

Note that our definition of the orbit cone agrees with [2, Definition 2.1] for the action of the torus T on the variety $\mathbb{V} // G$.

3. THE CASE OF $SO(V)$

Consider $G = SO(V)$, $\dim V \geq 3$. Let (\cdot, \cdot) be a non-degenerate symmetric bilinear form on V preserved by $SO(V)$. Since V is G -isomorphic to its dual V^* , we can assume that $m_2 = 0$, $m = m_1$, and $\mathbb{V} = V^m$. Let us construct the GIT-fan for the diagonal action $SO(V)$ on the variety $\mathbb{P}(V)^m$.

The algebra of invariants for the action $SO(V)$ on \mathbb{V} is generated by $u_{ij} = (v_i, v_j)$, where $(v_1, \dots, v_m) \in V^m$ [5, § 9.3]. Here the multidegrees are $f_{ii} := (0, \dots, 0, \underbrace{2}_i, 0, \dots, 0)$ and

$f_{ij} := (0, \dots, 0, \underbrace{1}_i, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0)$, $i, j = 1, \dots, m$. The morphism $\pi : \mathbb{V} \rightarrow \mathbb{V} // G$ sends (v_1, \dots, v_m) to the symmetric matrix $((v_i, v_j))_{i,j=1}^m$.

The GIT-fan is contained in \mathbb{Q}^m . Let x_1, \dots, x_m be the coordinates in this space.

Clearly, the weight cone Ω generated by f_{ij} is given by inequalities

$$x_i \geq 0, \quad i = 1, \dots, m.$$

Proposition 2. *Each $(m-1)$ -dimensional orbit cone lies in some hyperplane*

$$(1) \quad \sum_{i \in I} x_i = \sum_{j \in J} x_j,$$

where $I, J \subset \{1, \dots, m\}$, $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$.

Proof. The torus $T = (\mathbb{C}^\times)^m$ acts on V^m : $t \circ (v_1, \dots, v_m) = (t_1 v_1, \dots, t_m v_m)$. Then $t \circ (v_i, v_j) = t_i t_j (v_i, v_j)$. The orbit cone associated to $v = (v_1, \dots, v_m)$ is $(m-1)$ -dimensional if and only if the stabilizer $T_{\pi(v)}$ of the point $\pi(v)$ is one-dimensional.

Consider a graph Γ_v with the set of vertices $\{v_1, \dots, v_m\}$. By definition, v_i and v_j are joined by an edge in Γ_v if and only if $(v_i, v_j) \neq 0$. Assume that $(v_i, v_j) \neq 0$. Then any $t \in T_{\pi(v)}$ satisfies $t_i = t_j^{-1}$.

Let $\Gamma_v = \Gamma_1 \sqcup \dots \sqcup \Gamma_l$ be the decomposition into connected components. If Γ_k contains a cycle of odd length or a loop (type A), then $t_i^2 = 1$ for all $v_i \in \Gamma_k$ and $t \in T_{\pi(v)}$. In other case (type B), it is possible to divide the set of vertices of Γ_k into two subsets. For a point of the first subset $t_i = s_k$ holds, and for a point of the second subset we have $t_i = (s_k)^{-1}$, where $s_k \in \mathbb{C}^\times$. The stabilizer is one-dimensional if and only if there is only one component of type B in the graph Γ_v . Denote by I and J the sets of vertices in the first and the second subsets of this component. The weight f_{ij} lies in $\omega(v)$ if and only if $i \in I, j \in J$ or $j \in I, i \in J$. Hence the orbit cone is contained in hyperplane (1). \square

It follows from Proposition 2 that if dimension of the orbit cone $\omega(v)$ is less than $m - 1$, then $\omega(v)$ lies in the intersection of some $(m - 1)$ -dimensional orbit cones. Thus two points are GIT-equivalent if and only if they lie in the same m -dimensional and $(m - 1)$ -dimensional orbit cones.

Theorem 2. *For the diagonal action of the group $SO(V)$ on the variety $\mathbb{P}(V)^m$ the GIT-fan is obtained by cutting of the cone*

$$\Omega = \{(x_1, \dots, x_m) \mid x_i \geq 0\}$$

by hyperplanes

$$(1) \quad \sum_{i \in I} x_i = \sum_{j \in J} x_j,$$

where $I, J \subset \{1, \dots, m\}$, $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$.

Proof. It is sufficient to find all $(m - 1)$ -dimensional orbit cones. It follows from Proposition 2 that we should only prove that the intersection of each hyperplane (1) with the cone Ω is the orbit cone for some point v .

Let $v_k = (1, i, 0, \dots, 0)$ for $k \in I$, $v_j = (1, -i, 0, \dots, 0)$ for $j \in J$, and $v_l = (0, 0, 1, 0, \dots, 0)$ for $l \notin I \cup J$. (Here $i^2 = -1$). The orbit cone associated to v is generated by the weights f_{kj} ($k \in I, j \in J$) and f_{ll} ($l \notin I \cup J$). Hence $\omega(v)$ is $(m - 1)$ -dimensional and lies in hyperplane (1). It is easy to check that the rays $\langle f_{kj} \rangle$ ($k \in I, j \in J$) and $\langle f_{ll} \rangle$ ($l \notin I \cup J$) are precisely the edges of the intersection of Ω with hyperplane (1). This completes the proof of Theorem 2. \square

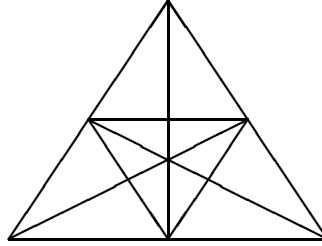
Example. Consider the action of SO_3 on the space $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3$.

The weight cone is the cone $\Omega = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. The GIT-fan is obtained by cutting of the cone Ω by hyperplanes

$$x_1 = x_2, \quad x_1 = x_3, \quad x_2 = x_3,$$

$$x_1 + x_2 = x_3, \quad x_1 + x_3 = x_2, \quad x_2 + x_3 = x_1.$$

The intersection of the GIT-fan with the hyperplane $x_1 + x_2 + x_3 = 1$ looks like:



There are 33 classes of GIT-equivalence: 12 classes are three-dimensional, 21 classes are two-dimensional, and 10 classes are one-dimensional.

4. THE CASE OF $SL(V)$

Consider $G = SL(V)$. Let us construct the GIT-fan for the diagonal action $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$.

The algebra of invariants $\mathbb{C}[\mathbb{V}]^{SL(V)}$, where $\mathbb{V} = V^{m_1} \oplus (V^*)^{m_2}$, is generated by $\det(v_{i_1}, \dots, v_{i_n})$, $\det(l_{j_1}, \dots, l_{j_n})$, and $l_j(v_i)$, where $(v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2}) \in V^{m_1} \oplus (V^*)^{m_2}$ [5, § 9.3]. Here the multidegrees are

$$f_{i_1 \dots i_n} = (\alpha_{i_1 \dots i_n}^1, \dots, \alpha_{i_1 \dots i_n}^{m_1}, \underbrace{0, \dots, 0}_{m_2}), \quad g_{j_1 \dots j_n} = (\underbrace{0, \dots, 0}_{m_1}, \beta_{j_1 \dots j_n}^1, \dots, \beta_{j_1 \dots j_n}^{m_2}),$$

$$\text{and } h_{ij} = (\varepsilon_{ij}^1, \dots, \varepsilon_{ij}^{m_1}, \delta_{ij}^1, \dots, \delta_{ij}^{m_2}),$$

where

$$\alpha_{i_1 \dots i_n}^{i_1} = \dots = \alpha_{i_1 \dots i_n}^{i_n} = \beta_{j_1 \dots j_n}^{j_1} = \dots = \beta_{j_1 \dots j_n}^{j_n} = \varepsilon_{ij}^i = \delta_{ij}^j = 1,$$

and other numbers are zero.

The GIT-fan is contained in \mathbb{Q}^m . Let $x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}$ be the coordinates in this space.

First, suppose that $m_1 \geq n$ or $m_2 \geq n$.

Proposition 3. *Each $(m-1)$ -dimensional orbit cone lies in one of the hyperplanes*

$$(2) \quad x_i = 0, i = 1, \dots, m_1,$$

$$(3) \quad y_j = 0, j = 1, \dots, m_2,$$

$$(4) \quad x_1 + \dots + x_{m_1} = y_1 + \dots + y_{m_2},$$

$$(5) \quad (n-k) \sum_{i \in I} x_i - k \sum_{i \notin I} x_i = (n-k) \sum_{j \in J} y_j - k \sum_{j \notin J} y_j,$$

where $1 \leq k \leq n-1$, $I \subset \{1, \dots, m_1\}$, $J \subset \{1, \dots, m_2\}$, and either $k \leq |I| \leq m_1 - n + k$ or $k \leq |J| \leq m_2 - n + k$.

Proof. If there is a zero vector or a zero function in the set $\{v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2}\}$, then the orbit cone associated to $v = (v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2})$ lies in hyperplane of type (2) or (3). Further, we assume that all the components of v are nonzero.

The torus $T = (\mathbb{C}^\times)^m$ acts on $V^{m_1} \oplus (V^*)^{m_2}$ as above. Then for any $t \in T$ we have

$$t \circ l_j(v_i) = t_i s_j l_j(v_i),$$

$$t \circ \det(v_{i_1}, \dots, v_{i_n}) = t_{i_1} \dots t_{i_n} \det(v_{i_1}, \dots, v_{i_n}),$$

$$t \circ \det(l_{j_1}, \dots, l_{j_n}) = s_{j_1} \dots s_{j_n} \det(l_{j_1}, \dots, l_{j_n}),$$

and the orbit cone associated to v is $(m-1)$ -dimensional if and only if the stabilizer $T_{\pi(v)}$ of the point $\pi(v)$ is one-dimensional.

Consider a graph Γ_v with $\{v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2}\}$ as the set of vertices. By definition, v_i and l_j are joined by an edge in Γ_v if and only if $l_j(v_i) \neq 0$. If vertices $v_{i_1}, v_{i_2}, l_{j_1}, l_{j_2}$ lie in the same connected component of the graph Γ_v , then any $t \in T_{\pi(v)}$ satisfies $t_{i_1} = t_{i_2}$, $s_{j_1} = s_{j_2}$, $t_{i_1} = s_{j_1}^{-1}$.

Case 1: $\dim\langle v_1, \dots, v_{m_1} \rangle < n$, $\dim\langle l_1, \dots, l_{m_2} \rangle < n$.

In this case all the determinants are zero. If the stabilizer $T_{\pi(v)}$ is one-dimensional, then the graph Γ_v is connected. The orbit cone is generated by the weights $\{h_{ij}\}$. Their span is $(m-1)$ -dimensional and lies in the hyperplane $x_1 + \dots + x_{m_1} = y_1 + \dots + y_{m_2}$. Hence the orbit cone lies in hyperplane of type (4).

Case 2: $\dim\langle v_1, \dots, v_{m_1} \rangle = n$, $\dim\langle l_1, \dots, l_{m_2} \rangle = n$.

Suppose $\det(v_{i_1}, \dots, v_{i_n}) \neq 0$. Then v_{i_1}, \dots, v_{i_n} is a basis of V . For any $t \in T_{\pi(v)}$ the equation $t_{i_1} \dots t_{i_n} = 1$ holds. If v_{i_1} occurs in the decomposition of v_i with respect to the basis v_{i_1}, \dots, v_{i_n} , then $\det(v_i, v_{i_2}, \dots, v_{i_n}) \neq 0$ and $t_i t_{i_2} \dots t_{i_n} = t_{i_1} t_{i_2} \dots t_{i_n} = 1$. Hence $t_i = t_{i_1}$. Similarly consider other v_{i_j} , where $j = 2, \dots, n$. Thus the space V decomposes into the sum $V = V_{k_1} \oplus \dots \oplus V_{k_r}$. The torus T multiplies any V_{k_l} by \bar{t}_l ($\bar{t}_l = t_{i_j}$ for some j). In the same way V^* decomposes into the sum $V^* = W_{\tilde{k}_1} \oplus \dots \oplus W_{\tilde{k}_q}$, the torus T acts on any $W_{\tilde{k}_l}$ as multiplication by \bar{s}_l . Thus any element $t \in T_{\pi(v)}$ satisfies the conditions $(\bar{t}_1)^{\dim V_{k_1}} \dots (\bar{t}_r)^{\dim V_{k_r}} = 1$ and $(\bar{s}_1)^{\dim W_{\tilde{k}_1}} \dots (\bar{s}_q)^{\dim W_{\tilde{k}_q}} = 1$.

Consider a new graph Γ'_v with $V_{k_1}, \dots, V_{k_r}, W_{\tilde{k}_1}, \dots, W_{\tilde{k}_q}$ as the set of vertices. The vertices V_k and $W_{\tilde{k}}$ are joined by an edge in Γ'_v if and only if there exist $v_i \in V_k$ and $l_j \in W_{\tilde{k}}$ such that $l_j(v_i) \neq 0$.

Denote by H_1, \dots, H_p the connected components of the graph Γ'_v . Let

$$V'_i = \bigoplus_{V_k \in H_i} V_k, \quad W'_i = \bigoplus_{W_{\tilde{k}} \in H_i} W_{\tilde{k}}.$$

Then $T_{\pi(v)}$ multiples V'_i by t'_i and W'_i by $(t'_i)^{-1}$. The stabilizer is given by the equations

$$(t'_1)^{\dim V'_1} \dots (t'_p)^{\dim V'_p} = 1,$$

$$(t'_1)^{\dim W'_1} \dots (t'_p)^{\dim W'_p} = 1.$$

By construction of H_i , all linear functions of W'_i vanish at all vectors of V'_j for $i \neq j$, hence $\dim W'_i \leq \dim V'_i$. But $\sum_{i=1}^p \dim W'_i = \sum_{i=1}^p \dim V'_i$, hence $\dim W'_i = \dim V'_i$.

Thus the stabilizer is given by the equation

$$(t'_1)^{\dim V'_1} \dots (t'_p)^{\dim V'_p} = 1,$$

and is one-dimensional if and only if $p = 2$.

So if the orbit cone associated to $v = (v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2})$ is $(m-1)$ -dimensional, then $V = V_1 \oplus V_2$, $V^* = W_1 \oplus W_2$, $\dim V_1 = \dim W_1 = k$, $1 \leq k \leq n-1$; any vector v_i lies in V_1 or in V_2 ; any linear function l_j lies in W_1 or in W_2 ; any linear function from W_j is zero on any vector from V_i for $i \neq j$.

Let I be the set of numbers of vectors v_i from V_1 , J be the set of numbers of linear functions l_j from W_1 . Then the orbit cone associated to v lies in hyperplane given by the equation

$$(n-k) \sum_{i \in I} x_i - k \sum_{i \notin I} x_i = (n-k) \sum_{j \in J} y_j - k \sum_{j \notin J} y_j.$$

Here the inequalities $k \leq |I| \leq m_1 - n + k$ and $k \leq |J| \leq m_2 - n + k$ are satisfied.

Case 3: $\dim \langle v_1, \dots, v_{m_1} \rangle = n$, $\dim \langle l_1, \dots, l_{m_2} \rangle < n$ or $\dim \langle v_1, \dots, v_{m_1} \rangle < n$, $\dim \langle l_1, \dots, l_{m_2} \rangle = n$. In this case we have one equation on the stabilizer and the graph Γ_v should have two connected components. We obtain hyperplanes given by equations (5). \square

Proposition 4. *The weight cone Ω is given by inequalities*

$$(6) \quad x_l \geq 0, \quad l = 1, \dots, m_1, \quad y_p \geq 0, \quad p = 1, \dots, m_2,$$

$$(n-k) \left(\sum_{j=1}^{m_2} y_j - \sum_{i \in I} x_i \right) + k \sum_{i \notin I} x_i \geq 0, \quad (n-k) \left(\sum_{i=1}^{m_1} x_i - \sum_{j \in J} y_j \right) + k \sum_{j \notin J} y_j \geq 0,$$

where $1 \leq k \leq n-1$, $I \subset \{1, \dots, m_1\}$, $J \subset \{1, \dots, m_2\}$, $|I| = |J| = k$.

Proof. It is sufficient to find hyperplanes (2)–(5) which contain facets of the cone Ω . It is clear that hyperplanes (2) and (3) do. Since the weights $\{f_{i_1 \dots i_n}\}$ and $\{g_{j_1 \dots j_n}\}$ lie on different sides of hyperplane (4), this hyperplane intersects the interior of the cone Ω .

Consider equation (5). First suppose that $0 < |I| < m_1$, $0 < |J| < m_2$. Let $i_1 \in I$, $i_2 \notin I$, $j_1 \in J$, $j_2 \notin J$. The weights $h_{i_1 j_2}$ and $h_{i_2 j_1}$ lie on different sides from the hyperplane. Now let $|J| = m_2$. If $|I| > k$, then there exist numbers $i_1, \dots, i_{k+1} \in I$, $i_{k+2}, \dots, i_n \notin I$. Weights $f_{i_1 \dots i_n}$ and $g_{j_1 \dots j_n}$ lie on different sides from the hyperplane. For $|I| = k$ we obtain inequalities (6). The cases $|I| = 0, m_1$ and $|J| = 0$ are analyzed similarly. \square

It follows from the proof of Proposition 4 that hyperplanes (4) and (5) intersect the interior of the cone Ω . Further, if dimension of the orbit cone $\omega(v)$ is less than $m-1$, then $\omega(v)$ lies in the intersection of some $(m-1)$ -dimensional orbit cones. Thus two points are GIT-equivalent if and only if they lie in the same m -dimensional and $(m-1)$ -dimensional orbit cones.

Theorem 3. For the diagonal action of the group $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$, where m_1 or m_2 does not exceed $n = \dim V$, the GIT-fan is obtained by cutting of the cone Ω given by inequalities (6) by hyperplanes

$$(4) \quad x_1 + \dots + x_{m_1} = y_1 + \dots + y_{m_2},$$

$$(5) \quad (n-k) \sum_{i \in I} x_i - k \sum_{i \notin I} x_i = (n-k) \sum_{j \in J} y_j - k \sum_{j \notin J} y_j,$$

where $1 \leq k \leq n-1$, $I \subset \{1, \dots, m_1\}$, $J \subset \{1, \dots, m_2\}$, $k \leq |I| \leq m_1 - n + k$ or $k \leq |J| \leq m_2 - n + k$.

Proof. Let us find all $(m-1)$ -dimensional orbit cones. It follows from Proposition 4 that we should only prove that the intersection Π of any hyperplane of type (4) or (5) with the cone Ω is the orbit cone for some point v .

For the case of hyperplane (4) let $v = (e_1, \dots, e_1, e^1, \dots, e^1)$. The orbit cone associated to v lies in this hyperplane and its dimension equals $m-1$. Note that the inequalities

$$(n-k) \left(\sum_{j=1}^{m_2} y_j - \sum_{i \in I} x_i \right) + k \sum_{i \notin I} x_i \geq 0$$

$$(n-k) \left(\sum_{i=1}^{m_1} x_i - \sum_{j \in J} y_j \right) + k \sum_{j \notin J} y_j \geq 0,$$

for the points of hyperplane (4) become

$$\sum_{i \notin I} x_i \geq 0 \quad \sum_{j \notin J} y_j \geq 0.$$

Hence the cone Π is the intersection of hyperplane (4) with the positive ortant. Finally, the weights which generate the orbit cone $\omega(v)$ and edges of the cone Π lie on the same rays.

Denote by H the hyperplane (5). Without loss of generality it can be assumed that equation (5) is of the form

$$(n-k) \sum_{i=1}^{|I|} x_i - k \sum_{i=|I|+1}^{m_1} x_i = (n-k) \sum_{j=1}^{|J|} y_j - k \sum_{j=|J|+1}^{m_2} y_j,$$

where $k \leq |I| \leq m_1 - n + k$.

The orbit cone $\omega(v)$ associated to the point

$$v = (e_1, \dots, e_k, \underbrace{e_k, \dots, e_k}_{|I|-k}, e_{k+1}, \dots, e_n, \underbrace{e_n, \dots, e_n}_{m_1+k-|I|-n},$$

$$\underbrace{e^1 + \dots + e^k, \dots, e^1 + \dots + e^k}_{|J|}, \underbrace{e^{k+1} + \dots + e^n, \dots, e^{k+1} + \dots + e^n}_{m_2-|J|})$$

is $(m-1)$ -dimensional and lies in hyperplane H . It is easy to check that this cone lies in Π . It remains to prove the converse implication.

Let $A = (x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \in \Pi$. Then

$$A = (x_1 - \sum_{j=2}^{|J|} y_j - \alpha) h_{11} + \sum_{i=2}^k (x_i - \alpha) h_{i1} + \sum_{i=k+1}^{|I|} x_i h_{i1} + (x_{|I|+1} - \sum_{j=|J|+2}^{m_2} y_j - \alpha) h_{|I|+1|J|+1} +$$

$$+ \sum_{i=|I|+2}^{|I|+n-k} (x_i - \alpha) h_{i|J|+1} + \sum_{i=|I|+n-k+1}^{m_1} x_i h_{i|J|+1} + \sum_{j=2}^{|J|} y_j h_{1j} + \sum_{j=|J|+2}^{m_2} y_j h_{|I|+1j} + \alpha f_{1\dots k|I|+1\dots|I|+n-k},$$

where $\alpha = \frac{1}{k}(\sum_{i=1}^{|I|} x_i - \sum_{j=1}^{|J|} y_j)$. It follows from inequalities (6), that coefficients of this decomposition are positive. Therefore $A \in \omega(v)$, and Π lies in $\omega(v)$. Hence Π coincides with $\omega(v)$. This completes the proof of Theorem 3. \square

Now let $m_1 < n$ and $m_2 < n$. In this case the weight cone is $(m-1)$ -dimensional.

Theorem 4. *For the diagonal action of the group $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$, where $m_1, m_2 < n = \dim V$, the GIT-fan is obtained by cutting of the cone*

$$\Omega = \{(x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}) \mid x_1 + \dots + x_{m_1} = y_1 + \dots + y_{m_2}; x_i, y_j \geq 0\}$$

by hyperplanes

$$(7) \quad \sum_{i \in I} x_i = \sum_{j \in J} y_j,$$

where $I \subset \{1, \dots, m_1\}$, $J \subset \{1, \dots, m_2\}$, $I \neq \emptyset, \{1, \dots, m_1\}$, $J \neq \emptyset, \{1, \dots, m_2\}$.

Proof. In this case the weight cone is generated by the weights $\{h_{ij}\}$. It is clear that the weight cone is contained in the cone Ω . On the other hand, edges of the cone Ω are precisely the generators of the weight cone.

We need to find all $(m-2)$ -dimensional orbit cones. As above let us construct the graph Γ_v for any vector v . The stabilizer $T_{\pi(v)}$ should be of dimension two. Hence the graph Γ_v has two connected components. In this case the orbit cone $\omega(v)$ is contained in the intersection of the weight cone Ω with hyperplane (7), where I and J are sets of numbers: $i \in I$ and $j \in J$ if v_i and l_j lie in the first connected component of the graph Γ_v . Finally it is necessary to prove that intersection of the cone Ω with hyperplane (7) is the orbit cone associated to some vector v . For this let the vector v be $(v_1, \dots, v_{m_1}, l_1, \dots, l_{m_2})$, where $v_i = e_1$, $l_j = e^1$, if $i \in I, j \in J$ and $v_i = e_2, l_j = e^2$, if $i \notin I, j \notin J$. This completes the proof of Theorem 4. \square

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